Multiplicity of Limit Cycle Attractors in Coupled Heteroclinic Cycles

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Abstract

A square lattice distribution of coupled oscillators that have heteroclinic cycle attractors is studied. In this system, we find a novel type of patterns that is spatially disordered and periodic in time. These patterns are limit cycle attractors in the ambient phase space (i.e. not chaotic) and many limit cycles exist dividing the phase space as their basins. The patterns are constructed with a local law of difference of phases between the oscillators. The number of patterns grows exponentially with increasing of the number of oscillators.

In the recent decades, coupled oscillators have attracted much attention. They are adopted as models for rhythmic or chaotic behaver of biological and other complex systems[1, 2, 3]. Moreover, coupled oscillators themselves are thought as the important classes to investigate in high dimensional dynamical systems[2, 4, 5]. For the analytical simplicity, limit cycles of a normal form type or phase oscillators have been frequently studied to replace detailed dynamics. While they have led to much development in understanding of high dimensional dynamical systems, there still remains rich phenomena that cannot be described with such typical oscillators.

Now, we present a new class of coupled oscillators in which each oscillator has a heteroclinic cycle attractor[6] instead of a limit cycle or a phase oscillator. Being different from those typical oscillators, an oscillator with a heteroclinic cycle attractor has no characteristic time scale by the following reason. A heteroclinic cycle is constructed with some saddle fixed points and heteroclinic orbits that connect the fixed points cyclically. When an orbit approaches to a heteroclinic cycle attractor, it stays long in the neighborhood of fixed points and moves along with heteroclinic orbits quickly. The length of the staying time grows exponentially for each oscillation while the moving speed between fixed points has little change. Then the period of oscillation gets exponentially longer and the system has no characteristic time scale. This property of heteroclinic cycles can bring quite complex structures such as the coexistence of

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infinitely many attractors[7]. Comparing with other oscillators with natural frequencies, this class of systems will give another plentiful phenomena and make our understandings of high dimensional dynamical systems richer.

In a general dynamical systems, heteroclinic cycles are always structurally unstable[6]. However, if a system has certain constraints or symmetries, they can make invariant sets in the phase space and a robust heteroclinic cycle can exist in the invariant sets. As a system with such constraints, we adopt a replicator system[7, 8, 9, 10, 11] for each oscillator, given by

$$\frac{dx_i}{dt} = x_i\Big(\sum_{j=1}^{4} a_{ij}x_j - \sum_{j,k=1}^{4} a_{jk}x_jx_k\Big), \qquad i = 1,\ldots,4 \tag{1}$$

$$\sum_{i=1}^{4} x_i = 1, \qquad 0 \le x_i \le 1 \tag{2}$$

with the parameter matrix

$$(a_{ij}) = \begin{pmatrix} 0 & -2 & -1 & 1 \\ 1 & 0 & -2 & -1 \\ -1 & 1 & 0 & -2 \\ -2 & -1 & 1 & 0 \end{pmatrix}. \tag{3}$$

With the constraints (2), the phase space of the replicator system with 4 components is restricted in a tetrahedron with $\boldsymbol{x} = (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)$ for the vertices(Fig.1). Vertices, edges ($\{\boldsymbol{x}|x_i+x_j = 1,\ x_k = x_l = 0\}$) and surfaces ($\{\boldsymbol{x}|x_i + x_j + x_k = 1,\ x_l = 0\}$) of the tetrahedron become invariant sets. In particular, vertices are always fixed points with any parameter matrix. Using the parameter matrix (3), an attracting heteroclinic cycle is constructed with the 4 vertices as saddle fixed points and the cyclically connecting edges as heteroclinic orbits(Fig.1).

In our model, the replicator systems are distributed on a square lattice and coupled between nearest neighbors diffusively. The equations are given as

$$\begin{aligned} \frac{dx_i}{dt}^{(u,v)} &= x_i^{(u,v)}\Big(\sum_{j=1}^{4} a_{ij}x_j^{(u,v)} - \sum_{j,k=1}^{4} a_{jk}x_j^{(u,v)}x_k^{(u,v)}\Big) \\ &+ D(x_i^{(u-1,v)} + x_i^{(u,v-1)} + x_i^{(u+1,v)} + x_i^{(u,v+1)} - 4x_i^{(u,v)}), \\ &\qquad i = 1,\ldots,4. \end{aligned} \tag{4}$$

where (u,v) is a site index that stands for a location of an oscillator and D is a diffusion constant between adjoining sites. Free boundary condition is employed. Since there is no non-diagonal diffusion term in the coupling method and the diffusion constant of each component has the same positive value, the spatially uniform oscillation never become unstable.

A replicator system is a model for a ecological system or a chemical reaction network of self-catalyzing molecules. Therefore the situation of our system can occur in a spotted ecological system with diffusion (ex. a system on trees of an orchard) or a population dynamics of self-catalyzing proteins in cells.

Before we state our main results for a system on 2-dimensional arrays we mention briefly about our preliminary results on systems with 1-dimensional lattices for comparison. If we choose free boundary condition for 1-dimensional systems (number of oscillators ≥ 2) all replicators are synchronized and only the spatially uniform oscillation become the attractor. Since all replicators are not chaotic and we choose the simple diffusive coupling as the interactions between oscillators mentioned above, this result is easily acceptable. If a system has periodic boundary (number of oscillators ≥ 3) with a certain small diffusion constant, traveling waves as well as the spatially uniform oscillation become attractors.

In this letter, we report the discovery of a novel class of patterns that are spatially disordered but periodic in time (Fig.2,3). From different initial conditions, a large variety of patterns are observed (Fig.2). Fig.3 shows that these disordered patterns are stable limit cycle attractors in the ambient phase space. Hence, these patterns are exactly recurrent and not chaotic. The different patterns correspond to the different limit cycles and they divide the phase space as their basins. After a system approaches sufficiently close to an attractor, frequency of each oscillator synchronizes with the limit cycle(Fig.3). This means that the frequencies of all oscillators are entrained while the phases of oscillations keep difference. The disordered patterns are observed in the weak coupling range ($D \leq 0.01$), while rotating spiral patterns of a well known type are seen with a certain large diffusion constant ($D \simeq 0.1$).

To Understand the structures of the disordered patterns, we note the local phase differences among oscillators. One can project the phase space of each replicator system on a 2-dimensional plane (α-β plane),

$$\begin{aligned} \alpha^{(u,v)} &= x_1^{(u,v)} - x_3^{(u,v)} \\ \beta^{(u,v)} &= x_2^{(u,v)} - x_4^{(u,v)} \end{aligned} \quad . \tag{5}$$

The vertices are projected on $(\alpha, \beta) = (1,0), (0,1), (-1,0), (0,-1)$ respectively. We plot orbit-points ($\boldsymbol{x}^{(u,v)}(t)$) of all oscillators on the α-β plane and connect between coupled oscillators with lines.

Typical snapshots after systems approach sufficiently close to attractors are shown in Fig.4. Comparing with Fig.4-a with a large D , Fig.4-b displays 2 features as local laws among oscillators. First, all orbit-points stay close to the heteroclinic cycle. Second, distances between coupling lines and the origin of the α-β plane are not smaller than a certain positive value. They are observed in every single step of numerical integrations. Therefore, the relation on locations of orbit-points for every two neighboring oscillators never become anti-phase. Considering these features, we conclude that the orbit-point arrangement of every set of the 4 oscillators that makes a unit square of the lattice (represented by dual-lattice point) is classified into 2 types; orbit-points of 4 oscillators surround the center (type-A) or not (type-B). Fig.5 shows configurations of the types.

To investigate the difference of the types in detail, let us consider a system within 2×2 oscillators as the simplest case. This system is thought as a system with 4 oscillators on 1-dimensional periodic array. Therefore, the situation

of type-A corresponds to the traveling wave solution. It is an attractor when the diffusion constant is smaller than about 0.01. Because phases of oscillators are different from each other, the diffusion effect prevents the orbit-points approaching to the heteroclinic cycle and the system keeps an uniform frequency. In contrast, if the system is arranged in type-B as the initial condition, oscillators are synchronized each other and the uniform oscillation approaches to the heteroclinic cycle. Therefore this system need to be arranged as type-A for the oscillation with an uniform frequency.

Returning to consider a large system, we note only on the type-As which generate oscillation with an uniform frequency. In such system, the type-As distributed on dual-lattice points can be regarded as another type of vortices whose properties are different from those of the spiral pattern in Fig.4-a. Therefore the disordered patterns are understood as coexistences of many vortices and the distributions of vortices on dual-lattice points would be important on characterization of the patterns.

To make our characterization of the patterns clearer and more systematic, we note another property of this system. As mentioned above, the diffusion effect keep orbit-points from approaching to heteroclinic cycle. Thus smaller diffusion constant makes orbit-points stay closer to the heteroclinic cycle and longer in neighborhoods of fixed points. Therefore, the time in a orbit-point passing along with a heteroclinic orbit become negligible and we see all orbit-points stay neighborhoods of the fixed points in typical snap-shots. With such small diffusion constant, we can systematically generate all possible snap-shots as initial conditions by choosing all orbit-points of replicators in neighborhoods of any fixed points.

Now, we choose the procedure for algorithmic generation of possible patterns as follows (i) distribute vortices (type-As) on the dual-lattice points without conflicts, (ii) select a initial condition orbit-points of oscillators from neighborhoods of 4 fixed points to satisfy condition (i). We check all possible patterns for 3×3 and 4×4 oscillators systems numerically and find that every candidate which satisfies condition (i) have at least one corresponding attractor and most of them have only one attractor. It means that almost all patterns are uniquely characterized by the distributions of vortices. Therefore we can estimate the number of attracting patterns with condition (i). Fig.6 shows the estimate that is algorithmically counted by computer. Since the condition (i) is a combination problem, the number of patterns grows exponentially with the increase of the number of oscillators.

In this letter, we have reported spatially disordered oscillating patterns in the coupled heteroclinic cycles. They have been shown periodic in time and not chaotic. The local law of differences of oscillators in the patterns and the estimation of the number of stable patterns have been also reported. In our forthcoming paper[12], we will analyze periods and stability of the patterns in detail.

Remarkable point of this system is that there is no obvious effect for violations from spatially uniform oscillation such as chaotic nature of each oscillator, variety of frequencies or anomalous coupling. Therefore the uniform oscillation

is always an attractor. Nevertheless, disordered patterns can exist as coexistence of vortices which structure is different from the well-known spiral pattern. This type of disordered oscillating patterns has not been found in any other coupled oscillators. However, other types of spatially disordered stable patterns have been reported in some class of dynamical systems[13, 14, 15, 16]. The relations among these patterns and ours will be discussed in the paper[12].

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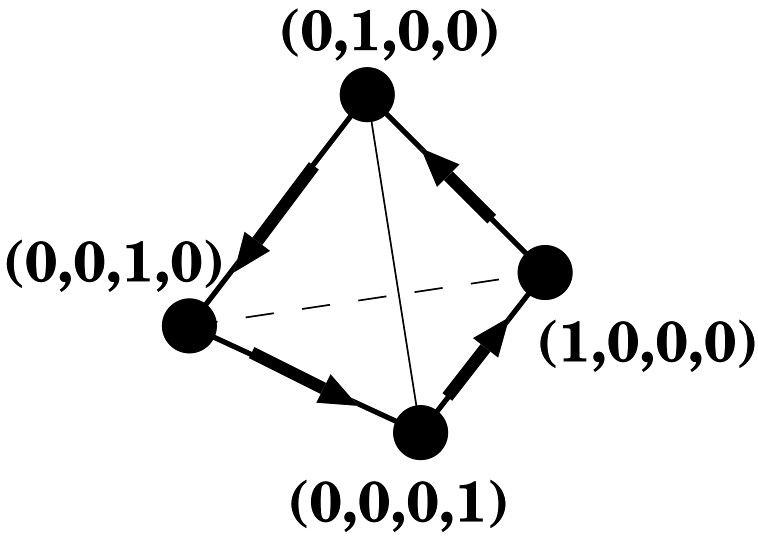

Figure 1: A phase space of replicator system with 4 components and heteroclinic cycle. Filled circles are saddle fixed points and Thick solid lines with arrows are heteroclinic orbits.

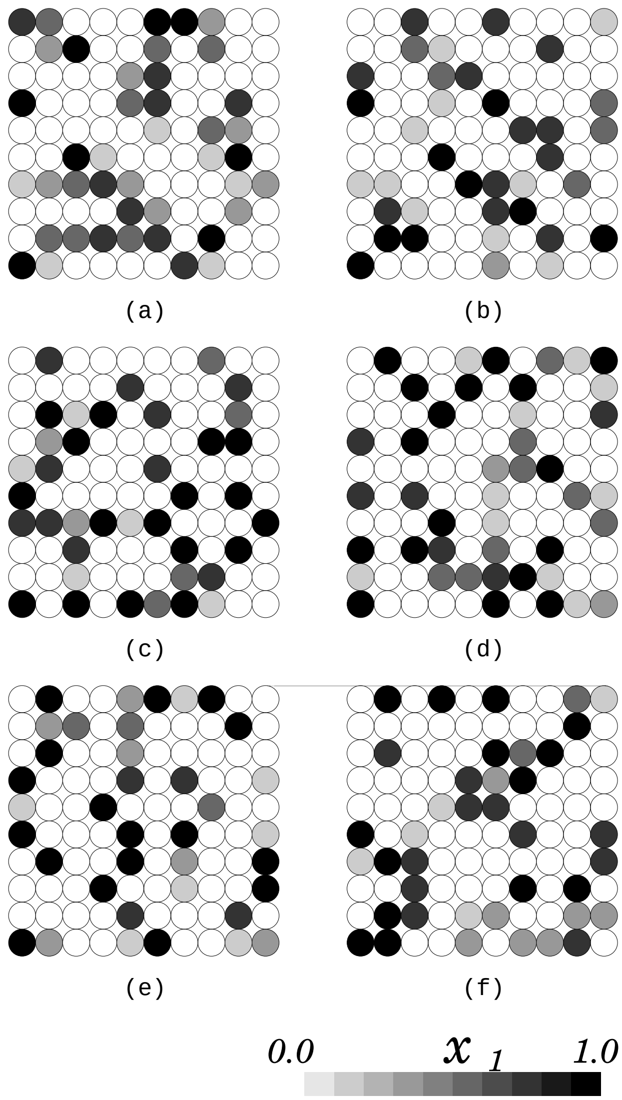

Figure 2: 6 snapshots of distributions of $x_1^{(u,v)}$ with a gray scale, 10×10 sites, $D = 10^{-4}$. (a)$\sim$(e) are generated from different initial conditions and taken the shots when the $(1,1)$-oscillators (the bottom left site) come to a same phase.

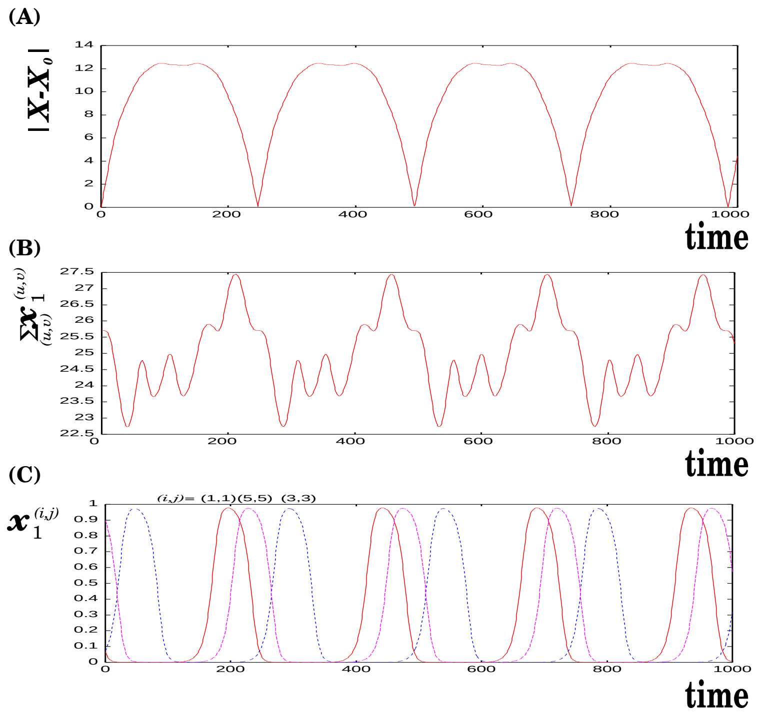

Figure 3: Time series data of Fig.1-(e). (A): Distance between $\boldsymbol{X}_0$ and $\boldsymbol{X}$ in the ambient phase space. $\boldsymbol{X}_0 = (x_1^{(1,1)}(t_0), x_2^{(1,1)}(t_0), \ldots, x_3^{(10,10)}(t_0), x_4^{(10,10)}(t_0))$ is a point in a trajectory at a time t_0 (after approaching sufficiently close to an attractor) and $\boldsymbol{X}$ is the trajectory after the time. (B): Mean field oscillation of x_1. (C): Oscillation of x_1 at (1,1),(3,3) and (5,5) sites. The exact recurrence in (A) indicate that the disordered pattern forms a limit cycle in the ambient phase space. Comparing (A) and (C), the oscillators synchronize with the limit cycle.

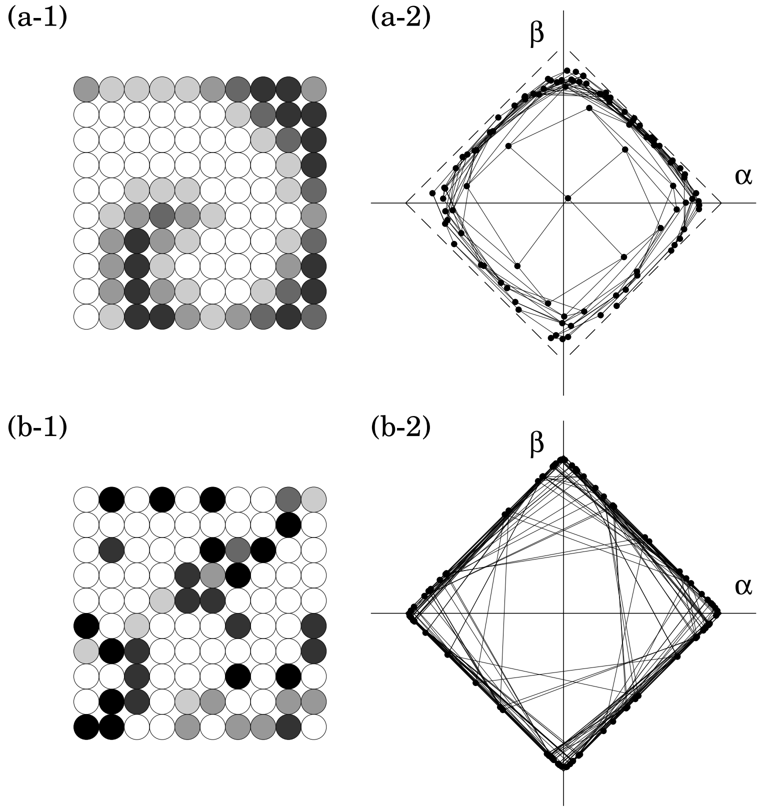

Figure 4: The snapshots of $x_1^{(u,v)}$ distributions (a-1,b-1)and corresponding phase differences in α-β plane (a-2,b-2). In (a-2,b-2), dots indicate projected orbit-points and the couplings between them are presented by lines. The broken line in (a-2) means projected heteroclinic cycle. $D = 10^{-2}$ in (a)-system and rotating spiral pattern is seen in (a-1). $D = 10^{-4}$ in (b)-system; our target pattern.

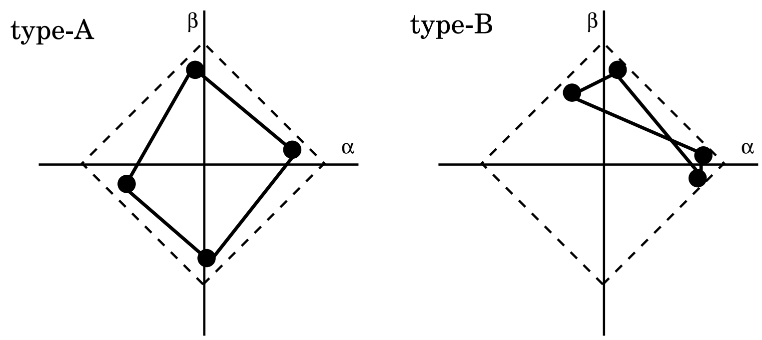

Figure 5: Typical configurations of a set of 4 oscillators placed in a unit square.

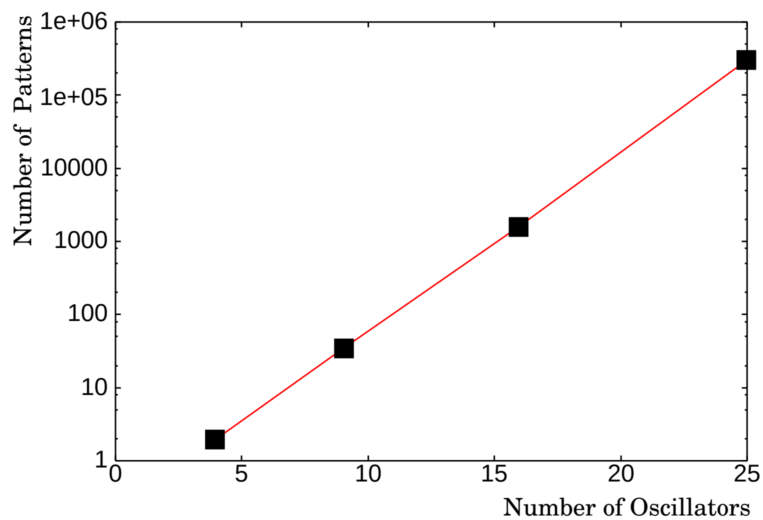

Figure 6: The estimates of the number of disordered patterns algorithmically calculated with condition (i). Regular square case (2×2, 3×3, 4×4 and 5×5 sites) are plotted.